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EE 364 - Spring 2026

Week 12

Supplemental Notes

Thrm: ("Weak" law of large numbers, WLLN) If i.i.d. $X_1, X_2, \ldots$ & $\sigma_X^2 < \infty$ (r.s.) $\bar{X}_n \xrightarrow{P} \mu_X$.

Pf: $\forall \varepsilon > 0$:

$$\lim_{n\to\infty} P[|\bar{X}_n - \mu_X| > \varepsilon] \overset{\text{thrm}}{=} \lim_{n\to\infty} P[|\bar{X}_n - E[\bar{X}_n]| > \varepsilon].$$

$$\overset{\text{C.I.}}{\leq} \lim_{n\to\infty} \frac{V[\bar{X}_n]}{\varepsilon^2}$$

$$= \lim_{n\to\infty} \frac{\sigma_X^2}{n\cdot\varepsilon^2}$$

$$= 0$$

QED

optional

Thrm: ("Strong" LLN, SLLN) $\bar{X}_n \xrightarrow{a} \mu_X$ if r.s.

Thrm: ("Mean-square" LLN, MSLLN) $\bar{X}_n \xrightarrow{m} \mu$ if r.s

Caveat: No such thing as a "law of averages"

Continuity Theorem: If g is continuous

$$\left\{ \begin{array}{l} X_n \xrightarrow{\circ} X \\ X_n \xrightarrow{p} X \\ X_n \xrightarrow{d} X \end{array} \right. \Longrightarrow \left\{ \begin{array}{l} g(X_n) \xrightarrow{\circ} g(X) \\ g(X_n) \xrightarrow{p} g(X) \\ g(X_n) \xrightarrow{d} g(X) \end{array} \right.$$

Note: $X_n \xrightarrow{m} X \not\Longrightarrow g(X_n) \xrightarrow{m} g(X)$

Ex: WLLN $\bar{X}_n \xrightarrow{p} \mu_x$ (iid, $\sigma^2 < \infty$)

put $g(x) = x^2$

$\therefore (\bar{X}_n)^2 \xrightarrow{p} \mu_x^2$

Issue spotting

Suppose $X_k \xrightarrow{p} X$

$$\lim_{n\to\infty} \sum_{k=1}^{n} g(X_k) = \lim_{n\to\infty} \overline{n \cdot g(X_n)} \xrightarrow{p} n\, E[g(X)]$$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} g(X_k) = \lim_{n\to\infty} \overline{g(X_n)} \xrightarrow{p} E[g(X)]$$

Monte Carlo Simulation

"Monte-Carlo" casino in Monaco

<u>2-types</u>

① iid M.C.

- generate (simulate) random samples (iid)
- Use LLN: $\overline{Y}_n \xrightarrow{mopd} (b-a)E_x[X]$

② "MCMC" Markov Chain MC (EE517)

- Correlated samples (from a Markov process)
  ↳ converge to equilibrium pdf π_e
- Problem with "Burn-in" and time to convergence
- Often use Gibbs Sampler (software: R or BUGS)
  ↳ Bayesian inference

Monte-Carlo estimation

1942 Manhattan project

1945 Trinity (U.S.)

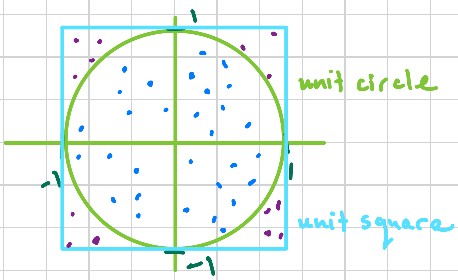
MC too slow for "the Super"

1949 Joe 1 (U.S.S.R.)

1952 Ivy-Mike (U.S.)

MCMC, "Metropolis"

Ex: estimate π



Area circle, $\pi \underbrace{r^2}_{=1} = \pi$

Area square, $\text{side}^2 = 4.$

$\therefore \quad \dfrac{A(c)}{A(s)} = \dfrac{\pi}{4}$

$\therefore \quad \pi = 4 \cdot \dfrac{A(c)}{A(s)} = 4 \cdot \dfrac{\text{blue dots}}{\text{total dots}}$

$\therefore$ use random sampling to compute ratio of areas.

$\therefore \quad P\{\underbrace{(x,y)}_{\text{sampled from unit square}} \text{ in circle}\} = P\{X^2 + Y^2 \le 1\} = \frac{\pi}{4}$

$\therefore$ So generate $U_1, U_2 \sim U[-1,1]$

then count # w/ $u_1^2 + u_2^2 \le 1$.

divide by total # of points.

Ex: Monte Carlo Integration

Q: how to evaluate $\int_a^b g(x)\,dx$ if g continuous but DON'T KNOW anti-derivative G?

A: IID MC estimate

① $\int_a^b g(x)\,dx = \frac{b-a}{b-a}\int_a^b g(x)\,dx$ $(a<b)$

$= (b-a)\int_a^b g(x)\cdot\left(\frac{1}{b-a}\right) g(x)$

$\overset{X\sim U[a,b]}{=} (b-a)\cdot E_X[g(X)].$

② Compute: $Y_k \triangleq (b-a)\cdot g(X_k)$ ← simulate

from iid draws $X_k \sim U[a,b]$

$\therefore \overline{Y_n} = \frac{1}{n}\sum_{k=1}^{n} Y_k = \frac{(b-a)}{n}\sum_{k=1}^{n} g(X_k)$

$\xrightarrow[P]{LLN} (b-a)\,E_X[g(X)] = \int_a^b g(x)\,dx$

$\therefore$ consistent.

$\therefore$ Monte Carlo approach:

$$\int_a^b g(x)\,dx = (b-a)\int_a^b \frac{g(x)}{b-a}\,dx \qquad \text{if } a<b.$$

$$= (b-a)\cdot E_x[g(x)] \quad \text{for r.v. } X\sim U[a,b].$$

$\therefore$ ① Draw iid $X_1,\dots,X_N \sim U[a,b]$ random number generator

② Put $Y_k = (b-a)\cdot g(X_k)$

③ $\overline{Y}_n \xrightarrow[\text{LLN}]{P,(a)} (b-a)\,E_x[g(X)] = \int_a^b g(x)\,dx$

Ex: estimate π : $4\int_0^1 \sqrt{1-x^2}\,dx = \pi$

put $a=0,\ b=1$; $g(x)=4\sqrt{1-x^2}$

$\therefore \int_0^1 g(x)\,dx = (1-0)\cdot\int_0^1 \frac{g(x)}{1-0}\,dx = E_x[g(X)]$ for r.v. $X\sim U[\overset{a}{0},\overset{b}{1}]$

① Draw iid $X_1,\dots,X_N \sim U[0,1]$

② Put $Y_k = 4\sqrt{1-X_k^2} = g(X_k)$

$\therefore \overline{Y}_N = \frac{1}{N}\sum_{k=1}^N Y_k = \frac{1}{N}\sum_{k=1}^N g(X_k) \xrightarrow[\text{LLN}]{P} E_x[g(X)]$

$$\boxed{= \int_0^1 4\sqrt{1-x^2}\,dx = \pi}$$

$$\boxed{m \longrightarrow p \longrightarrow d.}$$ ☆

"Mom Paid Dad" (mnemonic)

- Often easier to prove m (or p) than p (or d)

$$u \longrightarrow e \longrightarrow o \longrightarrow p \longrightarrow d.$$

"Uncle Ed Once Paid Dad"

- U implies all, even m.

Note: $e \not\longrightarrow m$

Thrm: $X_n \xrightarrow{m} X \longrightarrow X_n \xrightarrow{p} X$

Pf: Say $X_n \xrightarrow{m} X$: $\lim_{n\to\infty} E[(X_n - X)^2] = 0$. Pick $\varepsilon > 0$.

$$\lim_{n\to\infty} P[|X_n - X| > \varepsilon] = \lim_{n\to\infty} P[(X_n - X)^2 > \varepsilon^2]$$

$$\overset{M.I.}{\leq} \lim_{n\to\infty} \frac{E[(X_n - X)^2]}{\varepsilon^2}$$

$$= 0$$ (by hypothesis)

QED.

Ex: Say X_n are <u>similarly distributed</u> as $X_n \sim N(0, \frac{1}{n^2})$

($\therefore$ <u>not iid</u>)

redo at home w/ $n = 3, 4,$ etc.

Define $Y_n \stackrel{\Delta}{=} \sqrt{n} \cdot X_n$. Does $Y_1, Y_2, \dots$ converge in probability to 0?

Q: $Y_n \xrightarrow{P} 0$?

Method 1: $m \to p \to d$ approach $(m \to p)$

$$Y_n \xrightarrow{m} 0 \text{ iff } \lim_{n\to\infty} E[Y_n^2] = 0$$

$$\therefore \lim_{n\to\infty} E[Y_n^2] = \lim_{n\to\infty} E[n \cdot X_n^2] = \lim_{n\to\infty} n \cdot E[X_n^2]$$

$$= \lim_{n\to\infty} n \cdot V[X_n] \quad \text{since } X_n \sim N(\underline{0}, \tfrac{1}{n^2})$$

$$= \lim_{n\to\infty} \frac{n}{n^2} \quad \text{since } X_n \sim N(0, \underline{\tfrac{1}{n^2}})$$

$$= \lim_{n\to\infty} \frac{1}{n} = 0$$

$$\therefore Y_n \xrightarrow{P} 0 \text{ by } m \to p \quad (m \to p \to d)$$

Method 2: (later)

Method 3: Directly by definition of $Y_n \xrightarrow{P} Y$

pick $\varepsilon > 0$.

$$\therefore \lim_{n\to\infty} P[|Y_n - Y| > \varepsilon] = \lim_{n\to\infty} P[|\sqrt{n} X_n| > \varepsilon] = \lim_{n\to\infty} P[n X_n^2 > \varepsilon]$$

$$\stackrel{\text{M.I.}}{\leq} \lim_{n\to\infty} \frac{E[nX_n^2]}{\varepsilon^2} = \lim_{n\to\infty} \frac{n}{\varepsilon^2} E[X_n^2]$$

$$= \lim_{n\to\infty} \frac{n}{\varepsilon^2} \cdot V[X_n^2] \quad \text{since } X_n \sim N(\underline{0}, \tfrac{1}{n^2})$$

$$= \lim_{n\to\infty} \frac{n}{n^2 \varepsilon^2} \quad \text{since } X_n \sim N(0, \underline{\tfrac{1}{n^2}})$$

$$= 1/\varepsilon^2 \cdot \lim_{n\to\infty} \frac{1}{n} = 0$$

$$\therefore Y_n \xrightarrow{P} 0$$

Defn: Bias B of r.v. estimator $\hat{\Theta}_n$

$$B = \left| E[\hat{\Theta}_n] - \Theta \right|$$

Defn: $\hat{\Theta}_n$ is an unbiased estimator (u.B.) of Θ iff

$$E[\hat{\Theta}_n] = \Theta \quad \forall n.$$

Ex: $\overline{X}_n$ is unbiased because $E[\overline{X}_n] = \mu_x$

$\left(\therefore B = \left| E[\bar{X}_n] - \mu_x \right| = 0. \right)$

Defn: The mean-square error (MSE) of $\hat{\Theta}_n$: $\quad MSE(\hat{\Theta}_n) = E\left[(\hat{\Theta}_n - \theta)^2 \right]$

Thrm: ("Consistency") $\hat{\Theta}_n$ is consistent for Θ iff $\hat{\Theta}_n \xrightarrow{P} \Theta$

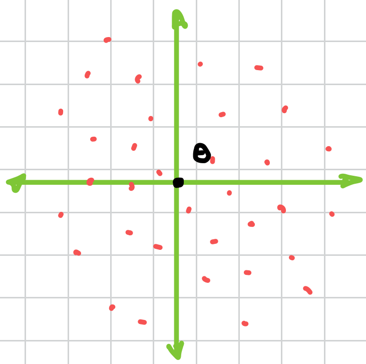
Ex: $\overline{X}_n$ is consistent for μ_x because $\overline{X}_n \xrightarrow{P} \mu_x$ (by WLLN)

Thm: ("Variance-Bias Decomposition")

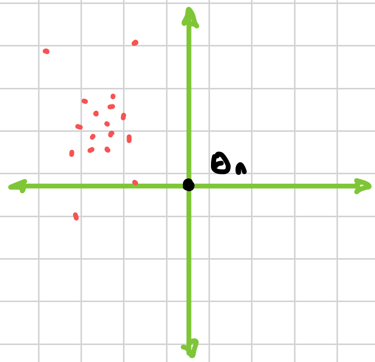
$$MSE[\hat{\theta}_n] = V[\hat{\theta}_n] + (E[\hat{\theta}_n] - \theta)^2$$

for $MSE[\hat{\theta}_n] = E[(\hat{\theta}_n - \theta_n)^2]$.

V vs B tradeoff



V-high, B-low



V-low, B-high.

Prf: $MSE[\hat{\theta}_n] = E[(\theta_n - \theta)^2]$

$$= E[((\theta_n - E[\theta_n]) + (E[\theta_n] - \theta))^2].$$

$$= \underbrace{E[(\hat{\theta}_n - E[\hat{\theta}_n])^2]}_{V[\hat{\theta}_n]} + \underbrace{(E[\theta_n] - \theta)^2}_{B^2}$$

$$+ \cancel{2(E[\hat{\theta}_n - \theta] - \theta)} \cdot \underbrace{E[\hat{\theta}_n - E[\hat{\theta}_n]]}_{E[\hat{\theta}_n] - E[\hat{\theta}_n] = 0}.$$

$$= V[\hat{\theta}_n] + Bias^2[\hat{\theta}_n].$$

QED

Always ask: Is $\hat{\theta}_n$ unbiased?
(if not is it asymptotically unbiased)

∴ Issue spotting sequence for m-p-d problems:

Use $MS = V + B^2$ and $m \longrightarrow p \longrightarrow d$

Q1: Is $\hat{\theta}_n$ unbiased (U.B.)? Asymptotically U.B.?

Q2: Is $\hat{\theta}_n$ consistent? (does $V \downarrow 0$)

If Q1 and Q2 yes: use $m \longrightarrow p \longrightarrow d$ as problem dictates

Ex: (cont) Say X_n are similarly distributed as $X_n \sim N(0, \frac{1}{n^2})$
(∴ not iid)

Define $Y_n \triangleq \sqrt{n} \cdot X_n$. Does $Y_1, Y_2, \dots$ converge in probability to 0? Q: $Y_n \xrightarrow{P} 0$?

Method 2: $MS = V + B^2$ and m-p-d.

since $X_n \sim N(0, \frac{1}{n^2})$

$$E[Y_n] = E[\sqrt{n} X_n] = \sqrt{n}\, E[X_n] = 0$$

∴ Y_n is U.B. for 0

since $X_n \sim N(0, \frac{1}{n^2})$

$$V[Y_n] = V[\sqrt{n} X_n] = n \cdot V[X_n] = \frac{n}{n^2} = \frac{1}{n}$$

$$\therefore \lim_{n\to\infty} MSE[Y_n] = \lim_{n\to\infty} \Big(\underbrace{V[Y_n]}_{=\frac{1}{n}} + (\underbrace{E[Y_n] - 0}_{=0 \ (\text{U.B.})})^2 \Big)$$

$$= \lim_{n\to\infty} \frac{1}{n} = 0$$

∴ $Y_n \xrightarrow{m} 0$ ∴ $Y_n \xrightarrow{p} 0$ by $m \longrightarrow p$.

Thrm: If iid $X_1, \dots, X_n$ and $E[|X|^4] < \infty$ then ($\therefore V[S^2]$ exists)

S_n^2 is unbiased for σ_x^2: $E[S_n^2] = \sigma_x^2$.

Prf:

$$S_n^2 = \frac{1}{n-1} \sum_{k=1}^{n} (X_k - \bar{X}_n)^2$$

$$= \frac{1}{n-1} \sum_{k=1}^{n} (X_k^2 + \bar{X}_n^2 - 2X_k\bar{X}_n)$$

$$= \frac{1}{n-1} \left(\sum_{k=1}^{n} X_k^2 + n \cdot \bar{X}_n^2 - 2\bar{X}_n \sum_{k=1}^{n} X_k \right)$$

$$= \frac{1}{n-1} \left(\sum_{k=1}^{n} X_k^2 + n \cdot \bar{X}_n^2 - 2n\bar{X}_n \overbrace{\left(\frac{1}{n} \sum_{k=1}^{n} X_k\right)}^{\bar{X}_n} \right)$$

$$= \frac{1}{n-1} \left(\sum_{k=1}^{n} X_k^2 + n \cdot \bar{X}_n^2 - 2n \cdot X_n^2 \right)$$

$$= \frac{1}{n-1} \left(\sum_{k=1}^{n} X_k^2 - n \cdot \bar{X}_n^2 \right)$$

$$\therefore E[S_n^2] \overset{\text{linearity}}{=} \frac{1}{n-1} \left(\sum_{k=1}^{n} E[X_k^2] - n \cdot E[\bar{X}_n^2] \right)$$

$$= \frac{1}{n-1} \left(\sum_{k=1}^{n} (V[X_k] + E^2[X_k]) - n \cdot (V[\bar{X}_n] + E^2[\bar{X}_n]) \right)$$

since $V[X] = E[X^2] - E^2[X]$ and finite 4^{th} moment $E[X^4]$.

$$= \frac{1}{n-1} \left(\sum_{k=1}^{n} (\sigma_x^2 + \mu_x^2) - n \cdot \left(\frac{\sigma_x^2}{n} + \mu_x^2 \right) \right)$$

$$= \frac{1}{n-1} \left(n \cdot \sigma_x^2 + \cancel{n \cdot \mu_x^2} - \sigma_x^2 - \cancel{n \cdot \mu_x^2} \right)$$

$$= \frac{n\sigma_x^2 - \sigma_x^2}{n-1}$$

$$= \sigma_x^2 \cancel{\frac{n-1}{n-1}}$$

$$= \sigma_x^2$$

QED

$\therefore E\left[\frac{1}{n}(X_k - \bar{X}_n)^2\right] = \frac{n-1}{n} S_n^2 \longrightarrow \sigma_x^2$

$\therefore$ asymptotically unbiased

(reason for $\frac{1}{n-1}$ vs $\frac{1}{n}$)

$\therefore V[S_n^2] < \infty$

Thrm: iid $X_1, X_2, \ldots$ & $\sigma_x^2 < \infty$ and $E[X^4] < \infty$

then $S_n^2 \xrightarrow{P} \sigma_x^2$. $\quad$ ($\therefore S_n^2$ is consistent for σ_x^2)

Prf:

$$S_n^2 = \frac{1}{n-1} \sum_{k=1}^{n} (X_k - \bar{X}_n)^2$$

from prev.

$$= \frac{1}{n-1} \sum_{k=1}^{n} (X_k^2 + \bar{X}_n^2 - 2 \cdot \bar{X}_n X_k)$$

$$= \frac{1}{n-1} \left(\sum_{k=1}^{n} X_k^2 + n \cdot \bar{X}_n^2 - 2 \cdot \bar{X}_n \sum_{k=1}^{n} X_k \right) \cdot \frac{n}{n}$$

$$= \frac{n}{n-1} \left(\frac{1}{n} \sum_{k=1}^{n} X_k^2 + X_n^2 - 2\bar{X}_n^2 \right)$$

$$= \frac{n}{n-1} \left(\frac{1}{n} \sum_{k=1}^{n} X_k^2 - \bar{X}_n^2 \right)$$

continuity theorem since $f(x) = x^2$ continuous.

$$\xrightarrow[\text{WLLN}]{P} 1 \cdot \left(E[X^2] - \mu_x^2 \right) = \sigma_x^2$$

QED.

$$(\bar{X}_n)^2 \longrightarrow (\mu_x)^2$$

Ex: iid $X_1, \ldots, X_n \sim N(\mu, \sigma^2)$. $\quad$ (1) Does $\bar{X}_n \xrightarrow{d} \mu_x$

(2) Does $S_n^2 \xrightarrow{P} \sigma^2$ (consistency)

(1) $$MSE[\bar{X}_n] \overset{V+B^2}{=} V[\bar{X}_n] + (E[\bar{X}_n] - \mu_x)^2$$

$$\overset{\text{iid}}{=} \frac{\sigma_x^2}{n} + \overset{\text{u.B.}}{0} \qquad \therefore \downarrow 0 \text{ as } n \uparrow \infty$$

$\therefore \bar{X}_n \xrightarrow{m} \mu \qquad \therefore \bar{X}_n \xrightarrow{d} \mu$ by m-p-d

(2) Fact: $V[S_n^2] = \frac{2\sigma^4}{n-1}$ if iid $N(\mu, \sigma^2)$

earlier: S_n^2 u.B. for σ^2

$$\therefore MSE[S_n^2] \overset{V+B^2}{=} V[S_n^2] + (\underbrace{E[S_n^2] - \sigma^2}_{=0 \text{ u.B.}})^2$$

$$= \frac{2\sigma^4}{n-1} \qquad \downarrow 0 \text{ as } n \uparrow \infty$$

$\therefore S_n^2 \xrightarrow{m} \sigma^2 \qquad \therefore S_n^2 \xrightarrow{p} \sigma^2$ by m-p-d

$\therefore$ Consistent

Q: Can you conclude $s_n \xrightarrow{p} \sigma_x$?

A: Yes. since $(\cdot)^2$ is continuous and so by continuity theorem.

But: $E[S_n] = E\left[\sqrt{S_n^2}\right] \leq \sqrt{E[S_n^2]} = \sqrt{\sigma_x^2} = \sigma_x$

($<$, strictly concave, by Jensen's ineq.)

$\therefore$ always biased (asymptotically unbiased)

optional

Thm: If $X_n \xrightarrow{p} X$ and $Y_n \xrightarrow{p} Y$ then $X_n + Y_n \xrightarrow{p} X+Y$.

Prf: Lemma: If $a>0$, $b>0$, and $c>0$ then $a+b>c \longrightarrow \left(a>\frac{c}{2} \text{ or } b>\frac{c}{2}\right)$

Prf: Say not. $a+b>c$ & $\sim\left(a>\frac{c}{2} \text{ OR } b>\frac{c}{2}\right)$

$\sim\left(a>\frac{c}{2} \text{ OR } b>\frac{c}{2}\right) \overset{\text{DeM}}{=} a \leq \frac{c}{2},\ b \leq \frac{c}{2}$

$\therefore a+b \leq \frac{c}{2}+\frac{c}{2} = c$ $\rightarrow\leftarrow$ contradiction

$\therefore a+b \leq c$ and $a+b>c$.

Pick $\varepsilon>0$:

$$\lim_{n\to\infty} P\left[\,|(X_n+Y_n)-(X+Y)|>\varepsilon\right]$$

$$= \lim_{n\to\infty} P\left[\,|(X_n-X)+(Y_n-Y)|>\varepsilon\right].$$

$$\leq \lim_{n\to\infty} P\left[\,|X_n-X|+|Y_n-Y|>\varepsilon\right]$$ since $|a+b| \leq |a|+|b|$ and P monotonicity

$$\overset{\text{Lemma}}{\leq} \lim_{n\to\infty} P\left[\,|X_n-X|>\tfrac{\varepsilon}{2} \text{ OR } |Y_n-Y|>\tfrac{\varepsilon}{2}\right]$$

$$\leq \underbrace{\lim_{n\to\infty} P\left[\,|X_n-X|>\tfrac{\varepsilon}{2}\right]}_{=0,\text{ since } X_n \xrightarrow{p} X} + \underbrace{\lim_{n\to\infty} P\left[\,|Y_n-Y|>\tfrac{\varepsilon}{2}\right]}_{=0,\text{ since } Y_n \xrightarrow{p} Y}.$$

$$= 0$$

$\therefore X_n+Y_n \xrightarrow{p} X+Y$

QED

optional

Thrm: If $X_n \xrightarrow{m} X$ and $Y_n \xrightarrow{m} Y$ then $X_n + Y_n \xrightarrow{m} X + Y$.

Prf:
$$\lim_{n\to\infty} E\left[\left((X_n+Y_n)-(X+Y)\right)^2\right]$$
$$= \lim_{n\to\infty} E\left[\left((X_n-X)+(Y_n-Y)\right)^2\right]$$
$$= \lim_{n\to\infty} E\left[(X_n-X)^2\right] + \lim_{n\to\infty} E\left[(Y_n-Y)^2\right] + \lim_{n\to\infty} 2\cdot E\left[(X_n-X)(Y_n-Y)\right]$$
$$= 0 + 0 + \lim_{n\to\infty} 2\cdot \text{"}\sigma_{XY}\text{"} \qquad \text{since } X_n \xrightarrow{m} X \text{ and } Y_n \xrightarrow{m} Y.$$
$$\leq 2\cdot \lim_{n\to\infty} \text{"}\sigma_X\text{"}\cdot\text{"}\sigma_Y\text{"} \qquad \text{by U.P. (Cauchy-Schwartz) } E[UV] \leq \sqrt{E[U^2]\cdot E[V^2]}$$
$$= 2\cdot \lim_{n\to\infty}\sqrt{E[(X_n-X)^2]}\cdot \lim_{n\to\infty}\sqrt{E[(Y_n-Y)^2]}$$
$$= 2\cdot\sqrt{\lim_{n\to\infty} E[(X_n-X)^2]}\cdot\sqrt{\lim_{n\to\infty} E[(Y_n-Y)^2]} \qquad \text{since } f(x)=x^2 \text{ continuous.}$$
$$= 0 \qquad \text{since } X_n \xrightarrow{m} X \text{ and } Y_n \xrightarrow{m} Y$$

$\therefore X_n + Y_n \xrightarrow{m} X + Y$

QED.

Caveat: $X_n + Y_n \xrightarrow{d} X + Y$ even if $X_n \xrightarrow{d} X$ and $Y_n \xrightarrow{d} Y$.

not in general

$P \longrightarrow d$ only for one sequence $X_1, X_2, X_3, \ldots$

in general